

A rotating incompressible perfect fluid space-time

Zoltán Perjés[†], Gyula Fodor[†], László Á. Gergely[†] and Mattias Marklund[‡]

[†] KFKI Research Institute for Particle and Nuclear Physics, Budapest 114, P.O.Box 49, H-1525 Hungary

[‡] Department of Plasma Physics, Umeå University, S-901 87 Umeå, Sweden

Abstract. A rigidly rotating incompressible perfect fluid solution of Einstein's gravitational equations is discussed. The Petrov type is D, and the metric admits a four-parameter isometry group. The Gaussian curvature of the instantaneous constant-pressure surfaces is positive and they have two ring-shaped cusps.

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Rotating perfect fluid solutions of the field equations of general relativity have been much sought after because of their importance in cosmology and in modeling relativistic stars. The purpose of this Letter is to present a perfect fluid space-time with the metric

$$ds^2 = \sin^4 \chi (dt + 2R \cos \theta d\varphi)^2 - 2 \sin^2 \chi R d\chi (dt + 2R \cos \theta d\varphi) - R^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) . \quad (1)$$

Here R is a constant. In relativistic units, with Einstein's gravitational constant chosen $k = 1$, the density is $\mu = 6/R^2$. The pressure is a function of the radial variable χ alone,

$$p = \frac{4}{R^2} \sin^{-2} \chi - \frac{6}{R^2} . \quad (2)$$

and the four-velocity has the form

$$u = \sin^{-2} \chi \frac{\partial}{\partial t} . \quad (3)$$

The pressure is positive in the domain $\chi \in (0, 0.95532)$. The acceleration is $\dot{u}_a dx^a = u_{a;b} u^b dx^a = -2 \cot \chi d\chi$, the shear vanishes and the vorticity vector is parallel to the acceleration. Hence, in the Collins-White classification [1] of perfect fluids, the type is *IIIAGii*. In Herlt's formalism [2], the metric belongs to class I. The weak energy condition is satisfied since $\mu + p > 0$.

To the authors' best knowledge, this is the first example of a rigidly rotating incompressible perfect fluid space-time. The solution was found by Ferwagner [3] and studied by one of the authors [4] when investigating locally rotationally symmetric perfect fluid space-times of class I in the Stewart-Ellis classification [5].

This solution of Einstein's gravitational equations can be obtained by a procedure similar to the one how the vacuum Kerr space-time arises from the vacuum Schwarzschild metric in the Eddington form $ds^2 = (1 - 2mr/\Sigma)dt^2 + 2dtdr - \Sigma(d\theta^2 + \sin^2 \theta d\varphi^2)$ (with

$\Sigma = r^2$) as the seed metric. One replaces the coordinate differentials $dt \rightarrow dt + a \sin^2 \theta d\varphi$ and $dr \rightarrow dr + a \sin^2 \theta d\varphi$, leaving Σ as a test potential. The vacuum Einstein equations then yield $\Sigma = r^2 + a^2 \cos^2 \theta$.

The interior Schwarzschild metric of a static perfect fluid

$$ds^2 = (A - \cos \chi)^2 dt^2 - R^2 [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)] \quad (4)$$

is brought to the more familiar form [6] by the coordinate transformation $\sin \chi = r/R$. The constant R determines the density, $\mu = 3/R^2$. The constant A is related to the radius r_1 of the matter ball by $A = (1 - r_1^2/R^2)^{1/2}$. The four-velocity is $u = (A - \cos \chi)^{-1} \partial/\partial t$.

There is no coordinate transformation bringing the interior Schwarzschild metric to a form corresponding to the Eddington metric. Nevertheless, we find that the simple metric $ds^2 = \sin^4 \chi dt^2 - 2R \sin^2 \chi d\chi dt - R^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)$ can be used for generating a perfect fluid space-time. The requisite substitution here is $dt \rightarrow dt + 2R \cos \theta d\varphi$ and $d\chi \rightarrow d\chi$.

Choosing a null frame along the principal directions of the curvature, we get the Weyl spinor component [7]

$$\Psi_2 = \frac{i \cot \chi - 1}{R^2}. \quad (5)$$

All other Weyl spinor components vanish, thus the Petrov type is D . There is a curvature singularity at the center $\chi = 0$. Closed time-like curves exist in the $\partial/\partial\varphi$ direction in a neighborhood \mathcal{N} of the axis $\theta = 0$, bounded by the surface $\tan \theta = 2 \sin \chi$.

The two-surfaces defined by constant values of the coordinates t and χ have a negative definite metric outside the domain \mathcal{N} and the pressure is constant on these surfaces. The Gaussian curvature

$$K = 4 \frac{1 + a^2}{a^2 R^2} \frac{\sin^4 \theta + a^2 \cos^4 \theta}{(\sin^2 \theta - a^2 \cos^2 \theta)^2}, \quad (6)$$

with the constant $a = 2 \sin \chi$, is positive and has two cusps on the boundary of \mathcal{N} .

Solution of the Killing equations reveals that the metric (1) contains four Killing vectors:

$$\begin{aligned} K_1 &= \partial/\partial t \\ K_2 &= \partial/\partial \varphi \\ K_3 &= 2R \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial t} + \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \\ K_4 &= 2R \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial t} - \sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi}. \end{aligned} \quad (7)$$

The symmetry group is $O(3) \times U(1)$. Thus the space-time (1) is locally rotationally symmetric and belongs to category (i) of the type-D perfect fluid classification scheme [8].

The singular behavior of the metric (1) on the symmetry axis can be removed on a semiaxis by introducing a new time coordinate $t' = t \pm 2R\varphi$. The physical significance

of the manifold on which the entire axis is regular, however, is debatable as is the case with the NUT space-time.

We obtain a solution of the Einstein equations with a cosmological term Λ by the substitution $\tilde{p} = p + \Lambda$ and $\tilde{\mu} = \mu - \Lambda$. In the cosmological solution, the domain of positive pressure can be extended or shrunk by a suitable choice of Λ .

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